The Mukai conjecture for log Fano manifolds

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THE MUKAI CONJECTURE FOR LOG FANO MANIFOLDS

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Abstract. For a log Fano manifold (X, D) with $D \neq 0$ and of the log Fano pseudoindex ≥ 2, we prove that the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D_1)$ of Picard groups is injective for any irreducible component $D_1 \subset D$. The strategy of our proof is to run a certain minimal model program and is similar to Casagrande's argument. As a corollary, we prove that the Mukai conjecture (resp. the generalized Mukai conjecture) implies the log Mukai conjecture (resp. the log generalized Mukai conjecture).

1. Introduction

Let X be a Fano manifold, that is, a complex smooth projective variety with $-K_X$ (the anticanonical divisor of X) ample. In this paper, we study the relation between the *Picard number* $\rho(X)$ of X and the *Fano index*

$$r(X) := \max\{r \in \mathbb{Z}_{>0} \mid -K_X \sim rL \text{ for some Cartier divisor } L \text{ on } X\}$$

or the *Fano pseudoindex*

$$\iota(X) := \min\{(-K_X \cdot C) \mid C \subset X \text{ rational curve}\}.$$

Especially, we are interested in the *Mukai conjecture* (resp. the *generalized Mukai conjecture*).

Conjecture 1.1 (Mukai conjecture [Muk88]). *Let X be an n-dimensional Fano manifold. Then the following inequality holds:*

$$\rho(X)(r(X) - 1) \leq n.$$

Moreover, equality holds if and only if X is isomorphic to the $\rho(X)$-th power of the $(r(X)-1)$-dimensional projective spaces $(\mathbb{P}^{r(X)-1})^{\rho(X)} (:= \prod^{\rho(X)} \mathbb{P}^{r(X)-1})$.

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Conjecture 1.2 (generalized Mukai conjecture [BCDD03]). *Let X be an n-dimensional Fano manifold. Then the following inequality holds:*

$$\rho(X)(\iota(X)-1) \leq n.$$

Moreover, equality holds if and only if X is isomorphic to the $\rho(X)$-th power of the $(\iota(X)-1)$-dimensional projective spaces $(\mathbb{P}^{\iota(X)-1})^{\rho(X)}$.

In this paper, we restate these conjectures as followsz.

Conjecture 1.3 (Conjecture M^n_ρ). *Fix $n, \rho \in \mathbb{Z}_{>0}$. Let X be an n-dimensional Fano manifold such that $\rho(X) \geq \rho$ and $r := r(X) \geq (n+\rho)/\rho$. Then $\rho(X) = \rho$, $r = (n+\rho)/\rho$ and $X \simeq (\mathbb{P}^{r-1})^\rho$ holds.*

Conjecture 1.4 (Conjecture GM^n_ρ). *Fix $n, \rho \in \mathbb{Z}_{>0}$. Let X be an n-dimensional Fano manifold such that $\rho(X) \geq \rho$ and $\iota := \iota(X) \geq (n+\rho)/\rho$. Then $\rho(X) = \rho$, $\iota = (n+\rho)/\rho$ and $X \simeq (\mathbb{P}^{\iota-1})^\rho$ holds.*

It is clear that the Mukai conjecture (resp. the generalized Mukai conjecture) is true if and only if Conjecture M^n_ρ (resp. Conjecture GM^n_ρ) is true for any $n, \rho \in \mathbb{Z}_{>0}$.

Recall that a *log Fano manifold* was originally introduced by [Mae86] (under a different name, logarithmic Fano variety) as a pair (X, D) of a complex smooth projective variety X and a reduced simple normal crossing divisor D on X such that $-(K_X + D)$ is ample. We are mainly interested in the relation between the Picard number $\rho(X)$ and the *log Fano index* $r(X, D)$ (resp. the *log Fano pseudoindex* $\iota(X, D)$). For the definitions of $r(X, D)$ and $\iota(X, D)$, see Definition 2.3 (similar to the definitions of the Fano index and the Fano pseudoindex for a Fano manifold). We addressed a special version of the log versions of the Mukai conjecture and the generalized Mukai conjecture in [Fjt12, Theorem 4.3]; we call them the *log Mukai conjecture* and the *log generalized Mukai conjecture* respectively. (In [Fjt12, Theorem 4.3], we proved Conjecture LGM^n_2.)

Conjecture 1.5 (log Mukai conjecture (LM^n_ρ)). *Fix $n, \rho \geq 2$. Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$ such that $\rho(X) \geq \rho$ and $r := r(X, D) \geq (n+\rho-1)/\rho$. Then $\rho(X) = \rho$, $r = (n+\rho-1)/\rho$ and (X, D) is isomorphic to the case of Type $(\rho, r; m_1, \ldots, m_{\rho-1})$ with $m_1 \ldots, m_{\rho-1} \geq 0$ in Example 4.1.*

Conjecture 1.6 (log generalized Mukai conjecture (LGM^n_ρ)). *Fix $n, \rho \geq 2$. Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$ such that $\rho(X) \geq \rho$ and $\iota := \iota(X, D) \geq (n+\rho-1)/\rho$. Then $\rho(X) = \rho$, $\iota = (n+\rho-1)/\rho$ and (X, D) is isomorphic to the case of Type $(\rho, \iota; m_1, \ldots, m_{\rho-1})$ with $m_1 \ldots, m_{\rho-1} \geq 0$ in Example 4.1.*

Remark 1.7. (i) Clearly, Conjecture GM^n_ρ (resp. Conjecture LGM^n_ρ) implies Conjecture M^n_ρ (resp. Conjecture LM^n_ρ) (see Remark 2.4).

(ii) Conjecture LM^n_ρ (resp. Conjecture LGM^n_ρ) explains that for any n-dimensional log Fano manifold (X, D) with $D \neq 0$, the following inequality holds:

$$\rho(X)(r(X,D)-1) \leq n-1 \quad \big(\text{resp. } \ \rho(X)(\iota(X,D)-1) \leq n-1\big),$$

and describes all the (X, D) for which the equality holds.

(iii) If an n-dimensional log Fano manifold (X, D) with $D \neq 0$ satisfies the inequality $\iota(X, D) \geq n$, then $\iota(X, D) = n$, $X \simeq \mathbb{P}^n$ and D is a hyperplane by [Fjt12, Proposition 4.1]. In particular, if (X, D) is a one-dimensional log Fano manifold with $D \neq 0$, then $X \simeq \mathbb{P}^1$ and D is a reduced one point. Thus the case $n = 1$ or $\rho = 1$ for Conjecture LGM^n_ρ (and also Conjecture GM^n_ρ) is rather trivial. That is why we only consider the case n, $\rho \geq 2$.

(iv) Conjecture GM^n_ρ has been considered by many people. Nowadays, it is known that Conjecture GM^n_ρ is true if $n \leq 5$ ([ACO04]) or $\rho \leq 3$ ([NO10]). See [ACO04], [NO10] and references therein.

(v) We note that Conjecture LGM^2_ρ is a straightforward consequence of the classification of 2-dimensional log Fano manifolds in [Mae86, §3], since the pair $(\mathbb{P}^2, \text{line})$ is the only case of log Fano pseudoindex > 1. See also [Fjt12, Proposition 4.1].

In this article, we obtain a fundamental property to compare $\mathrm{Pic}(X)$ and $\mathrm{Pic}(D)$ for a log Fano manifold (X, D).

Theorem 1.8 (see Theorem 3.8)**.** *Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$. Then one of the following holds:*

(1) *The restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective.*

(2) *X admits a $\mathbb{P}^1$-bundle structure $\pi\colon X \to Y$ for which D is a section. In particular, D is irreducible and isomorphic to Y (hence Y is an $(n-1)$-dimensional Fano manifold).*

Compare Theorem 1.8 and Casagrande's original result for Fano manifolds.

Theorem 1.9 ([Cas11, Theorem 1.2])**.** *Let X be a Fano manifold with $\iota(X) > 1$. Then one of the following holds:*

(i) *$\iota(X) = 2$ and X admits a $\mathbb{P}^1$-bundle structure $\phi\colon X \to Y$, where Y is a Fano manifold with $\iota(Y) > 1$.*

(ii) *For any prime divisor $D \subset X$, the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective.*

As a consequence of Theorem 1.8, for a log Fano manifold (X, D) with $\iota(X, D) \geq 2$ and $D \neq 0$, we get a comparison theorem of the Picard number of X and $D_1 \subset D$.

Corollary 1.10 (= Corollary 3.9 (1))**.** *Let (X, D) be a log Fano manifold with $\iota(X, D) \geq 2$ and $D \neq 0$. Then the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D_1)$ is injective for any irreducible component $D_1 \subset D$.*

To prove Theorem 1.8, we use the result of [BCHM10] that X is a *Mori dream space* (see [HK00] for the definition) for a log Fano manifold (X, D). We run a special $(-D)$-minimal model program (*MMP*, for short) and compare the cokernel of the homomorphism $\mathrm{N}_1(D) \to \mathrm{N}_1(X)$ in each step of the MMP. We can show that the dimension of the cokernel is constant by using arguments from [Cas09] and [Cas11].

As a corollary, we can show that the Mukai Conjecture (resp. the generalized Mukai Conjecture) implies the log Mukai Conjecture (resp. the log generalized Mukai Conjecture).

Theorem 1.11 (= Theorem 4.4)**.** *Fix n, $\rho \geq 2$. Conjectures $\mathrm{M}_\rho^{n'}$ for all $n' \leq n$ (resp. Conjectures $\mathrm{GM}_\rho^{n'}$ for all $n' \leq n$) imply Conjecture LM_ρ^{n+1} (resp. Conjecture LGM_ρ^{n+1}).*

Using this theorem, we obtain the following corollary immediately.

Corollary 1.12 (= Corollary 4.5 (1))**.** *Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$ such that $\rho(X) \geq 3$ and $\iota := \iota(X, D) \geq (n+2)/3 > 1$. Then $\iota = (n+2)/3$,*

$$X \simeq \mathbb{P}_{\mathbb{P}^{\iota-1}\times\mathbb{P}^{\iota-1}}(\mathcal{O}_{\mathbb{P}^{\iota-1}\times\mathbb{P}^{\iota-1}}^{\oplus\iota} \oplus \mathcal{O}_{\mathbb{P}^{\iota-1}\times\mathbb{P}^{\iota-1}}(m_1, m_2))$$

for some integers $m_1 \geq 0$ and $m_2 \geq 0$, and

$$D \simeq \mathbb{P}_{\mathbb{P}^{\iota-1}\times\mathbb{P}^{\iota-1}}(\mathcal{O}_{\mathbb{P}^{\iota-1}\times\mathbb{P}^{\iota-1}}^{\oplus\iota}),$$

where the embedding is obtained by the canonical projection under these isomorphisms.

Notation and terminology. We use the same notation as our previous paper [Fjt12].

We always work in the category of algebraic (separated and finite type) schemes over the complex number field $\mathbb{C}$. A *variety* means a connected and reduced algebraic scheme. For a variety X, the set of singular points on X is denoted by $\mathrm{Sing}(X)$.

For the theory of extremal contraction, we refer the readers to [KM98]. For a complete variety X, the Picard number of X is denoted by $\rho(X)$. For a complete variety X and a closed subscheme D on X, the image of the homomorphism $\mathrm{N}_1(D) \to \mathrm{N}_1(X)$ is denoted by $\mathrm{N}_1(D, X)$. For a smooth projective variety X and a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$,

$$l(R) := \min\{(-K_X \cdot C) \mid C \text{ is a rational curve with } [C] \in R\}$$

is called the *length* $l(R)$ of R. A rational curve $C \subset X$ with $[C] \in R$ and $(-K_X \cdot C) = l(R)$ is called *a minimal rational curve in* R.

For a morphism of algebraic schemes $f\colon X \to Y$, we define the *exceptional locus* $\mathrm{Exc}(f)$ *of* f by

$$\mathrm{Exc}(f) := \{x \in X \mid f \text{ is not isomorphism around } x\}.$$

For algebraic schemes $X_1, \dots, X_m$, the projection $\prod_{i=1}^m X_i \to \prod_{j=1}^k X_{i_j}$ is denoted by $p_{i_1,\dots,i_k}$ for any $1 \leq i_1 < \cdots < i_k \leq m$.

For an algebraic scheme X and a locally free sheaf $\mathcal{E}$ of finite rank on X, let $\mathbb{P}_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck and $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ be the tautological invertible sheaf. We usually denote the projection by $p\colon \mathbb{P}_X(\mathcal{E}) \to X$.

We write

$$\mathcal{O}_{\prod_{i=1}^s \mathbb{P}^{n_i}}(m_1, \dots, m_s)$$

on $\prod_{i=1}^s \mathbb{P}^{n_i}$ instead of $p_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(m_1) \otimes \cdots \otimes p_s^*\mathcal{O}_{\mathbb{P}^{n_s}}(m_s)$ for simplicity.

A morphism $f\colon X \to Y$ is called a $\mathbb{P}^r$*-bundle* if f is a smooth proper morphism and any closed fiber of f is (scheme-theoretically) isomorphic to $\mathbb{P}^r$.

For a variety X and a reduced Cartier divisor D on X, we often regard D as an algebraic scheme with the natural (reduced) scheme structure.

2. Log Fano manifolds

We recall the definitions and some properties of log Fano manifolds and snc Fano varieties quickly. For more informations, see [Fjt12, Section 2].

Definition 2.1. (i) A variety $\mathcal{X}$ is called an n-dimensional *simple normal crossing* (*snc*, for short) *Fano variety* if $\mathcal{X}$ is an equi-n-dimensional projective variety having normal crossing singularities (that is, the formal completion of the local ring $\mathcal{O}_{\mathcal{X},x}$ is isomorphic to

$$\mathbb{C}[[x_1, \dots, x_{n+1}]]/(x_1 \cdots x_k)$$

for some $1 \leq k \leq n+1$, for any closed point $x \in \mathcal{X}$), each irreducible component X of $\mathcal{X}$ is smooth and $\omega_{\mathcal{X}}^{\vee}$ (the dual of the dualizing sheaf) is ample.

(ii) An n-dimensional *log Fano manifold* is a pair (X, D) where X is an n-dimensional smooth projective variety and D is a reduced and simple normal crossing divisor on X (that is, D has normal crossing singularities and each irreducible component of D is smooth) such that $-(K_X + D)$ is ample, where K_X is the canonical divisor of X.

Remark 2.2. We remark that the notion of "log Fano manifold" is much stronger than a projective dlt pair (X, Δ) with X smooth and $-(K_X + \Delta)$ ample.

Definition 2.3. (i) Let $\mathcal{X}$ be an snc Fano variety. We define the *snc Fano index* $r(\mathcal{X})$ (resp. the *snc Fano pseudoindex* $\iota(\mathcal{X})$) of $\mathcal{X}$ as

$$\begin{aligned} r(\mathcal{X}) &:= \max\{r \in \mathbb{Z}_{>0} \mid \omega_{\mathcal{X}}^{\vee} \simeq \mathcal{L}^{\otimes r} \text{ for some } \mathcal{L} \in \operatorname{Pic}(\mathcal{X})\} \\ (\text{resp. } \iota(\mathcal{X}) &:= \min\{\deg_C(\omega_{\mathcal{X}}^{\vee}|_C) \mid C \subset \mathcal{X} \text{ rational curve}\}). \end{aligned}$$

(ii) Let (X, D) be a log Fano manifold. We define the *log Fano index* $r(X, D)$ (resp. the *log Fano pseudoindex* $\iota(X, D)$) of (X, D) as

$$\begin{aligned} r(X, D) &:= \max\{r \in \mathbb{Z}_{>0} \mid -(K_X + D) \sim rL \text{ for some Cartier divisor } L \text{ on } X\} \\ (\text{resp. } \iota(X, D) &:= \min\{(-(K_X + D) \cdot C) \mid C \subset X \text{ rational curve}\}). \end{aligned}$$

Remark 2.4. For an snc Fano variety $\mathcal{X}$, $\iota(\mathcal{X})$ is divisible by $r(\mathcal{X})$. For a log Fano manifold (X, D), $\iota(X, D)$ is divisible by $r(X, D)$.

Remark 2.5 ([Fjt12, Theorem 2.20 (1)])**.** Let (X, D) be a log Fano manifold. Then D is a (connected) snc Fano variety such that $r(D)$ is divisible by $r(X, D)$ and $\iota(D) \geq \iota(X, D)$.

Remark 2.6. Let $\mathcal{X}$ be an snc Fano variety and $\mathcal{X} = \bigcup_{1 \leq i \leq m} X_i$ be its decomposition into irreducible components. Then the pair

$$(X_I, D_I) := \left(\bigcap_{i \in I} X_i, \sum_{j \notin I} \left(X_j|_{(\bigcap_{i \in I} X_i)}\right)\right)$$

is a log Fano manifold such that $r(X_I, D_I)$ is divisible by $r(\mathcal{X})$ and $\iota(X_I, D_I) \geq \iota(\mathcal{X})$ holds for any subset $I \subset \{1, \ldots, m\}$. In particular, $Z := X_1 \cap \cdots \cap X_m$ is a (smooth and connected) Fano manifold such that $r(Z)$ is divisible by $r(\mathcal{X})$ and $\iota(Z) \geq \iota(\mathcal{X})$ holds.

Proof. We know that $\bigcap_{i \in I} X_i$ is a nonempty, connected and smooth variety by [Fjt12, Theorem 2.20 (2)]. Hence the assertion follows from the adjunction formula. $\square$

Now, we show several properties for log Fano manifolds and snc Fano varieties. See also [Fjt12].

Proposition 2.7 ([Fjt12, Proposition 2.8, Theorem 2.20 (2)])**.** *Let $\mathcal{X}$ be an n-dimensional snc Fano variety and $\mathcal{X} = \bigcup_{i=1}^{m} X_i$ be its decomposition into irreducible components. We also let X_{ij} be the (scheme theoretical) intersection $X_i \cap X_j$ for any $1 \leq i < j \leq m$. Then we have an exact sequence*

$$0 \to \operatorname{Pic}(\mathcal{X}) \xrightarrow{\eta} \bigoplus_{i=1}^{m} \operatorname{Pic}(X_i) \xrightarrow{\mu} \bigoplus_{1 \leq i < j \leq m} \operatorname{Pic}(X_{ij}),$$

where η is the restriction homomorphism and

$$\mu\Big((\mathcal{H}_i)_i\Big) := (\mathcal{H}_i|_{X_{ij}} \otimes \mathcal{H}_j^{\vee}|_{X_{ij}})_{i<j}.$$

Lemma 2.8 ([Mae86, Corollary 2.2, Lemma 2.3]). *Let (X, D) be a log Fano manifold, or more generally a projective dlt pair with $-(K_X + D)$ ample. Then $\operatorname{Pic}(X)$ is torsion free. Furthermore, the homomorphism*

$$\operatorname{Pic}(X) \to H^2(X^{\mathrm{an}}; \mathbb{Z})$$

is an isomorphism.

The following result is essential in this article.

Theorem 2.9 ([BCHM10, Corollary 1.3.2]). *If a projective pair (X, D) is $\mathbb{Q}$-factorial dlt with $-(K_X + D)$ ample, then the variety X is a Mori dream space. In particular, for a log Fano manifold (X, D), the variety X is a Mori dream space.*

3. Running a minimal model program

In this section, we consider a special minimal model program for a log Fano manifold, whose argument is similar to Casagrande's argument [Cas09, Cas11].

First, we recall a result of Ishii.

Lemma 3.1 ([Ish91, Lemma 1.1]). *Let Y be a projective variety with canonical singularities. Let $R \subset \overline{\mathrm{NE}}(Y)$ be a K_Y-negative extremal ray such that the contraction morphism $\pi\colon Y \to Z$ associated to R is of birational type, and let $E := \operatorname{Exc}(\pi)$. Assume that each fiber of the restriction morphism $\pi|_E\colon E \to \pi(E)$ to its image is of dimension one. Then each fiber of $\pi|_E$ is a union of smooth rational curves and $0 < (-K_Y \cdot l) \leq 1$ for a component l of a fiber of $\pi|_E$ which contains a Gorenstein point of Y.*

We recall that we can run a B-MMP for any $\mathbb{Q}$-divisor B for a Mori dream space.

Proposition 3.2 ([HK00, Proposition 1.11 (1)]). *Let X be a Mori dream space and B be a $\mathbb{Q}$-divisor on X. Then there exists a sequence of birational maps among normal, $\mathbb{Q}$-factorial and projective varieties*

$$X = X^0 \overset{\sigma_0}{\dashrightarrow} X^1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-1}}{\dashrightarrow} X^k$$

and a $\mathbb{Q}$-divisor B^i on X^i for any $0 \leq i \leq k$ such that

(i) *The birational map σ_i is decomposed into the following diagram*

$$\begin{tikzcd} X^i \arrow[rr, dashed, "\sigma_i"] \arrow[dr, "\pi_i"'] & & X^{i+1} \arrow[dl, "\pi_i^+"] \\ & Y^i & \end{tikzcd}$$

and B^i is the strict transform of B on X^i for any $0 \leq i \leq k-1$.

(ii) *The morphism* π_i *is the birational contraction morphism associated to an extremal ray* $R^i \subset \mathrm{NE}(X^i)$ *such that* $(B^i \cdot R^i) < 0$ *and* π_i^+ *is the flip of* π_i *(if* π_i *is small) or the identity morphism (if* π_i *is divisorial) for any* $0 \leq i \leq k-1$.

(iii) *Either* B^k *is nef on* X^k *or there exists a fiber type extremal contraction* $X^k \xrightarrow{\pi_k} Y^k$ *associated to the extremal ray* $R^k \subset \mathrm{NE}(X^k)$ *such that* $(B^k \cdot R^k) < 0$ *holds.*

We call this step by a B-minimal model program *(a* B-MMP*, for short).*

For a log Fano manifold (X, D), or more generally a projective $\mathbb{Q}$-factorial dlt pair (X, D) with $-(K_X + D)$ ample, the variety X is a Mori dream space by Theorem 2.9. Hence we can apply Proposition 3.2. Moreover, we can choose a B-MMP which is also a $(K_X + D)$-MMP. The proof is completely the same as that of [Cas11, Proposition 2.4] (replacing $-K_X$ with $-(K_X + D)$).

Proposition 3.3. *Let* (X, D) *be a projective,* $\mathbb{Q}$*-factorial dlt pair such that* $-(K_X + D)$ *is ample, and* B *be a* $\mathbb{Q}$*-divisor on* X*. Then we can choose a* B*-MMP which is also a* $(K_X + D)$*-MMP.*

We are in particular interested in the case where B is equal to $-D$.

Notation 3.4. Let (X, D) be a projective dlt pair such that $-(K_X + D)$ is ample. We assume that X is smooth and D is a nonzero, effective and reduced Cartier divisor. Let $D = \sum_{i=1}^m D_i$ be the decomposition of D into irreducible components. We consider a $(-D)$-MMP (as in Proposition 3.2) which is also a $(K_X + D)$-MMP as in Proposition 3.3 (we note that this is also a K_X-MMP). We set D_i^j such as the strict transform of D_i in X^j for any $1 \leq i \leq m$ and $0 \leq j \leq k$. Let $A^1 \subset X^1$ be the indeterminacy locus of σ_0^{-1}, and for $2 \leq j \leq k$, let $A^j \subset X^j$ be the union of the indeterminacy locus of σ_{j-1}^{-1} and the union of the strict transforms of all components of $A^{j-1} \subset X^{j-1}$ not contained in the exceptional locus of the birational map σ_{j-1}.

The next lemma is essentially established by Casagrande [Cas09]. For a proof, see [Cas09, Lemma 3.8].

Lemma 3.5 (cf. [Cas09, Lemma 3.8])**.** *Under Notation 3.4, we have the following properties:*

(1) *For any* $1 \leq j \leq k$*, the dimension of* A^j *is at most* $n-2$*,* $X^j \setminus A^j$ *is isomorphic to an open subscheme of* X *and*

$$\mathrm{Sing}(X^j) \subset A^j \subset D^j$$

holds. Moreover, $\dim A^j > 0$ *whenever* π_{j-1} *is small.*

(2) *For any $1 \leq j \leq k$, X^j has terminal singularities and the pair (X^j, D^j) is a $\mathbb{Q}$-factorial dlt pair. Moreover, if $C \subset X^j$ is an irreducible curve not contained in A^j and $C^0 \subset X$ its strict transform, we have*

$$(-(K_{X^j} + D^j) \cdot C) \geq (-(K_X + D) \cdot C^0),$$

with the strict inequality whenever $C \cap A^j \neq \emptyset$.

The next proposition is the key of this article.

Proposition 3.6 (see [Cas11, Lemma 2.6])**.** *Under Notation 3.4, we have the following properties:*

(1) *For any $0 \leq j \leq k$, the divisor D^j is nonzero effective. In particular, this MMP ends with a fiber type contraction. That is, there exists a fiber type extremal contraction $X^k \xrightarrow{\pi_k} Y^k$ associated to the extremal ray $R^k \subset \mathrm{NE}(X^k)$ such that $(D^k \cdot R^k) > 0$ and $((K_{X^k} + D^k) \cdot R^k) < 0$ holds. The restriction morphism $\pi_k|_{D^k} \colon D^k \to Y^k$ is surjective.*

(2) *The restriction morphism $\pi_j|_{D_i^j} \colon D_i^j \to \pi_j(D_i^j)$ to its image is an algebraic fiber space, that is, $(\pi_j|_{D_i^j})_*\mathcal{O}_{D_i^j} = \mathcal{O}_{\pi_j(D_i^j)}$, for any $1 \leq i \leq m$ and $0 \leq j \leq k$.*

(3) *There exists an irreducible curve $C^j \subset D^j$ such that $\pi_j(C^j)$ is a point for any $0 \leq j \leq k-1$.*

(4) *If the restriction morphism $\pi_k|_{D^k} \colon D^k \to Y^k$ is a finite morphism, then $k = 0$ and the morphism $(\pi_0 =)\pi_k \colon X^k \to Y^k$ is a $\mathbb{P}^1$-bundle and $(D =)D^k$ is a section of π_k.*

(5) *We set the log Fano pseudoindex $\iota(X, D)$ of the pair (X, D) as the minimum of the intersection number $(-(K_X + D) \cdot C)$, where C is a rational curve on X. If $\iota(X, D) \geq 2$, then $\dim Y^k \leq n - 2$ holds.*

Proof. (1) We prove by induction on j that D^j is a nonzero effective divisor on X for all $0 \leq j \leq k$. The case $j = 0$ is trivial. Assume that $j \geq 1$ and D^{j-1} is nonzero effective. We assume that D^j is not nonzero effective. Then D^{j-1} is a prime divisor and π_{j-1} is a divisorial contraction which contracts D^{j-1}, but this leads to a contradiction since $(D^{j-1} \cdot R^{j-1}) > 0$. Thus D^j is nonzero effective for any $0 \leq j \leq k$. Since D^k is nonzero effective, $-D^k$ cannot be nef. Therefore this MMP ends with a fiber type contraction. We also know that the restriction morphism $\pi_k|_{D^k} \colon D^k \to Y^k$ is surjective since any fiber and D^k intersect each other.

(2) It is enough to show that the homomorphism $(\pi_j)_*\mathcal{O}_{X^j} \to (\pi_j|_{D_i^j})_*\mathcal{O}_{D_i^j}$ is surjective. We know that the sequence

$$(\pi_j)_*\mathcal{O}_{X^j} \to (\pi_j|_{D_i^j})_*\mathcal{O}_{D_i^j} \to R^1(\pi_j)_*\mathcal{O}_{X^j}(-D_i^j)$$

is exact. Since the pair (X^j, D^j) is a $\mathbb{Q}$-factorial dlt pair by Lemma 3.5 (2), we know that the pair $(X^j, \sum_{i'\neq i} D^j_{i'})$ is also a dlt pair by [KM98, Corollary 2.39]. Since $-D^j_i - (K_{X^j} + \sum_{i'\neq i} D^j_{i'}) = -(K_{X^j} + D^j)$ is (π_j)-ample, we have $R^1(\pi_j)_*\mathcal{O}_{X^j}(-D^j_i) = 0$ by [Fjn09, Theorem 2.42]. Therefore the restriction morphism $\pi_j|_{D^j_i}\colon D^j_i \to \pi_j(D^j_i)$ to its image is an algebraic fiber space.

(3) Assume that the restriction morphism $\pi_j|_{D^j}\colon D^j \to Y^j$ is a finite morphism for some $0 \leq j \leq k-1$. Let F^j be an arbitrary nontrivial fiber of π_j. Then F^j and D^j intersect each other since $(D^j \cdot R^j) > 0$. If $\dim F^j \geq 2$, then $\dim(F^j \cap D^j) \geq 1$ since D^j is a $\mathbb{Q}$-Cartier divisor. This is a contradiction to the assumption that $\pi_j|_{D^j}$ is a finite morphism. Therefore $\dim F^j = 1$ for any nontrivial fiber of π_j. Let $l^j \subset F^j$ be an arbitrary irreducible component. Then $l^j \not\subset A^j$ since $A^j \subset D^j$ by Lemma 3.5 (1), and $(D^j \cdot l^j) > 0$ by the property $(D^j \cdot R^j) > 0$. Hence we can apply Lemma 3.1; we have $(-K_{X^j} \cdot l^j) \leq 1$. Let $l \subset X$ be the strict transform of $l^j \subset X^j$. Then

$$(-(K_X + D) \cdot l) \leq (-(K_{X^j} + D^j) \cdot l^j) = (-K_{X^j} \cdot l^j) - (D^j \cdot l^j) < 1$$

holds by Lemma 3.5 (2). This leads to a contradiction since $-(K_X+D)$ is an ample Cartier divisor. Therefore the restriction morphism $\pi_j|_{D^j}\colon D^j \to Y^j$ is not a finite morphism for any $0 \leq j \leq k-1$.

(4) We have $\dim Y^k = n-1$ by (1). If there exists a fiber $F^k \subset X^k$ of π_k such that $\dim F^k \geq 2$, then $\dim(D^k \cap F^k) \geq 1$ holds. This leads to a contradiction since $\pi_k|_{D^k}$ is a finite morphism. Thus any fiber of π_k is of dimension one. We can take a general smooth fiber $l^k \subset X^k$ of π_k such that $l^k \cap A^k = \emptyset$. Since $(-(K_{X^k} + D^k) \cdot R^k) > 0$, $(D^k \cdot R^k) > 0$ and $l^k \cap \mathrm{Sing}(X^k) = \emptyset$ (hence D^k and K_{X^k} is Cartier around l^k), we have $l^k \simeq \mathbb{P}^1$, $(-K_{X^k} \cdot l^k) = 2$ and $(D^k \cdot l^k) = 1$. We assume that $k \geq 1$. Then $A^k \neq \emptyset$ holds. Let $l^k_0 \subset X^k$ be a fiber of π_k such that $l^k_0 \cap A^k \neq \emptyset$ holds. We know that $(-(K_{X^k} + D^k) \cdot l^k_0) = 1$ by [Kol96, Theorem 1.3.17]. We note that any arbitrary irreducible component l^k_1 of l^k_0 satisfies $l^k_1 \not\subset A^k$ since $l^k \not\subset D^k$and $A^k \subset D^k$ holds by Lemma 3.5 (1). Let $l^k_1 \subset l^k_0$ be an irreducible component such that $l^k_1 \cap A^k \neq \emptyset$ holds, and let $l_1 \subset X$ be the strict transform of $l^k_1 \subset X^k$. Then we have

$$(-(K_X + D) \cdot l_1) < (-(K_{X^k} + D^k) \cdot l^k_1) \leq 1$$

by Lemma 3.1. However, this leads to a contradiction since $-(K_X + D)$ is an ample Cartier divisor. Hence $k = 0$ holds. Thus the morphism $\pi_0 = \pi_k\colon X \to Y^0$ has the property that $\dim F^0 = 1$ for any fiber of π_0, and for general fiber $l^0 \subset X$, we have $(-K_X \cdot l^0) = 2$ and $(D \cdot l^0) = 1$. Therefore π_0 is a $\mathbb{P}^1$-bundle and D is a section of π_0 by [Fjt87, Lemma 2.12].

(5) Assume that $\dim Y^k = n-1$. Then a general fiber $l^k \subset X^k$ of π^k satisfies the conditions, $l^k \cap A^k = \emptyset$, $l^k \simeq \mathbb{P}^1$ and $(-K_{X^k} \cdot l^k) = 2$ by the same argument of the proof of (4). Thus we have

$$(-(K_X + D) \cdot l) \leq (-(K_{X^k} + D^k) \cdot l^k) < 2,$$

where $l \subset X$ is the strict transform of $l^k \subset X^k$, by Lemma 3.1 and the property $(D^k \cdot l^k) > 0$. This contradict to the property $\iota(X, D) \geq 2$. Therefore $\dim Y^k \leq n-2$ holds. □

Corollary 3.7 (see [Cas09, Lemma 3.6])**.** *Under Notation 3.4, we have the following results:*

(1) *The equality* $\rho(X) - \dim \mathrm{N}_1(D, X) = \rho(X^j) - \dim \mathrm{N}_1(D^j, X^j)$ *holds for any* $0 \leq j \leq k$.

(2) *We have* $\rho(X) - \dim \mathrm{N}_1(D, X) = 0$ *or* 1. *If* $\rho(X) - \dim \mathrm{N}_1(D, X) = 1$, *then* $k = 0$, *the morphism* $\pi_0 \colon X \to Y^0$ *is a* $\mathbb{P}^1$*-bundle and* D *is a section of* π_0.

Proof. (1) We prove by induction on j that (1) holds. The case $j = 0$ is obvious. We consider the case $1 \leq j \leq k$. It is enough to show the equality $\rho(X^{j-1}) - \dim \mathrm{N}_1(D^{j-1}, X^{j-1}) = \rho(X^j) - \dim \mathrm{N}_1(D^j, X^j)$. We know that $\dim \mathrm{N}_1(\pi_{j-1}(D^{j-1}), Y^{j-1}) = \dim \mathrm{N}_1(D^{j-1}, X^{j-1}) - 1$ by Proposition 3.6 (3).

If π_{j-1} is small, then any curve in X^j that is contracted by π^+_{j-1} is in D^j since $-D^j$ is (π^+_{j-1})-ample. Hence $\dim \mathrm{N}_1(\pi_{j-1}(D^{j-1}), Y^{j-1}) = \dim \mathrm{N}_1(D^j, X^j) - 1$. Therefore $\rho(X^{j-1}) - \dim \mathrm{N}_1(D^{j-1}, X^{j-1}) = \rho(X^j) - \dim \mathrm{N}_1(D^j, X^j)$ holds since $\rho(X^{j-1}) = \rho(X^j)$.

If π_{j-1} is divisorial, then $\sigma_{j-1} = \pi_{j-1}$ and $\rho(X^j) = \rho(X^{j-1}) - 1$ holds. Therefore $\rho(X^{j-1}) - \dim \mathrm{N}_1(D^{j-1}, X^{j-1}) = \rho(X^j) - \dim \mathrm{N}_1(D^j, X^j)$ holds.

(2) The value $\rho(X^k) - \dim \mathrm{N}_1(D^k, X^k)$ is equal to 0 or 1 since the restriction morphism $\pi_k|_{D^k} \colon D^k \to Y^k$ is surjective and the dimension of the kernel of the surjection $(\pi_k)_* \colon \mathrm{N}_1(X^k) \to \mathrm{N}_1(Y^k)$ is one. If $\rho(X^k) - \dim \mathrm{N}_1(D^k, X^k) = 1$, then the restriction homomorphism $\mathrm{N}_1(D^k, X^k) \to \mathrm{N}_1(Y^k)$ is isomorphism. Thus any curve in D^k cannot be contracted. Hence the assertion holds by Proposition 3.6 (4). □

As an immediate corollary, we get the following theorem. We note that Theorem 1.8 is a direct consequence of Theorem 3.8.

Theorem 3.8. *Let* (X, D) *be a pair as in Notation 3.4. Then one of the following holds:*

(1) *The restriction homomorphism* $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ *is injective.*

(2) X *admits a* $\mathbb{P}^1$*-bundle structure* $\pi \colon X \to Y$ *and* D *is a section of* π. *In particular,* D *is irreducible and isomorphic to* Y *(hence* Y *is an* $(n-1)$*-dimensional Fano manifold and* $\iota(X, D) = 1$*).*

Proof. If $\rho(X) - \dim \mathrm{N}_1(D, X) = 1$, then (2) holds by Corollary 3.7 (2). If $\rho(X) - \dim \mathrm{N}_1(D, X) = 0$, then the homomorphism $\mathrm{N}_1(D) \to \mathrm{N}_1(X)$ is surjective. Hence the dual homomorphism $\mathrm{N}^1(X) \to \mathrm{N}^1(D)$ is injective. We know that the canonical homomorphism $\mathrm{Pic}(X) \to \mathrm{N}^1(X)$ is injective by Lemma 2.8, hence the homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective. $\square$

As a corollary of Theorem 1.8, we get the following property which is important to classify higher dimensional log Fano manifolds of log Fano pseudoindices ≥ 2.

Corollary 3.9. (1) *Let (X, D) be a log Fano manifold with $D \neq 0$ and $\iota(X, D) \geq 2$. Then the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D_1)$ is injective for any irreducible component $D_1 \subset D$.*

(2) *Let $\mathcal{X}$ be an snc Fano variety with $\iota(\mathcal{X}) \geq 2$. Then the restriction homomorphism $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_1)$ is injective for any irreducible component $X_1 \subset \mathcal{X}$.*

Proof. (1) We prove by induction on the dimension of X. If $\dim X = 2$, then the result follows from Remark 1.7 (v); we have $X \simeq \mathbb{P}^2$ and D is a hyperplane under the isomorphism.

We can assume that the assertion holds for any log Fano manifold (X', D') with $\dim X' = \dim X - 1$. If D is irreducible, then the assertion holds by Theorem 1.8 (1). Let $D = \sum_{i=1}^{m} D_i$ be the decomposition of D into irreducible components and let $D_{ij} := D_i \cap D_j$ for any $i \neq j$; we can assume $m \geq 2$. We assume that an invertible sheaf $\mathcal{H}$ on X satisfies $\mathcal{H}|_{D_1} \simeq \mathcal{O}_{D_1}$. It is enough to show that $\mathcal{H} \simeq \mathcal{O}_X$. We note that $(D_i, \sum_{j\neq i} D_{ij})$ is a log Fano manifold with $\iota(D_i, \sum_{j\neq i} D_{ij}) \geq 2$ for any $1 \leq i \leq m$. Hence the restriction homomorphism $\mathrm{Pic}(D_i) \to \mathrm{Pic}(D_{1i})$ is injective for any $2 \leq i \leq m$ by the induction step. Thus $\mathcal{H}|_{D_i} \simeq \mathcal{O}_{D_i}$ for any $1 \leq i \leq m$ since $(\mathcal{H}|_{D_i})|_{D_{1i}} = (\mathcal{H}|_{D_1})|_{D_{1i}} \simeq \mathcal{O}_{D_{1i}}$ and the injectivity of the homomorphism $\mathrm{Pic}(D_i) \to \mathrm{Pic}(D_{1i})$ for $2 \leq i \leq m$. Therefore $\mathcal{H}|_D \simeq \mathcal{O}_D$ by Proposition 2.7; we remark that D is an snc Fano variety. As a consequence, $\mathcal{H} \simeq \mathcal{O}_X$ holds by Theorem 1.8 (1).

(2) Let $\mathcal{X} = \bigcup_{i=1}^{m} X_i$ be the decomposition of $\mathcal{X}$ into irreducible components and let $X_{ij} := X_i \cap X_j$ for any $i \neq j$; we can assume that $m \geq 2$. We assume that an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ satisfies $\mathcal{L}|_{X_1} \simeq \mathcal{O}_{X_1}$. It is enough to show that $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X}}$. We note that $(X_i, \sum_{j\neq i} X_{ij})$ is a log Fano manifold with $\iota(X_i, \sum_{j\neq i} X_{ij}) \geq 2$. Thus the restriction homomorphism $\mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{1i})$ is injective for any $2 \leq i \leq m$ by (1). We deduce that $\mathcal{L}|_{X_i} \simeq \mathcal{O}_{X_i}$ since $(\mathcal{L}|_{X_i})|_{X_{1i}} = (\mathcal{L}|_{X_1})|_{X_{1i}} \simeq \mathcal{O}_{X_{1i}}$ and the injectivity of the homomorphism $\mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{1i})$ for any $2 \leq i \leq m$. Therefore we have $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X}}$ by Proposition 2.7. $\square$

From Theorem 1.8, we have a following technical lemma. This lemma is essential to prove Theorem 3.11.

Lemma 3.10. *Let $\mathcal{X}$ be an snc Fano variety and $\mathcal{X} = \bigcup_{1\leq i\leq m} X_i$ be its decomposition into irreducible components. We assume that $m \geq 2$. Then the natural homomorphism*

$$\operatorname{Pic}(\mathcal{X}) \to \bigoplus_{I\subset\{1,\ldots,m\},|I|=k} \operatorname{Pic}\left(\bigcap_{i\in I} X_i\right)$$

induced by restrictions is injective for any $1 \leq k \leq m-1$.

Proof. We prove Lemma 3.10 by induction on k. If $k = 1$, the assertion is trivial by Proposition 2.7. We assume that $1 < k \leq m-1$. Assume that $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ satisfies the condition that $\mathcal{L}|_{\bigcap_{i\in J} X_i}$ is trivial for any $J \subset \{1,\ldots,m\}$ with $|J| = k$. Take any subset $I \subset \{1,\ldots,m\}$ with $|I| = k-1$. It is enough to show that $\mathcal{L}|_{\bigcap_{i\in I} X_i}$ is trivial (by the inductive assumption). We know that the pair

$$\left(\bigcap_{i\in I} X_i, \sum_{j\notin I}\left(X_j|_{(\bigcap_{i\in I} X_i)}\right)\right)$$

is a log Fano manifold by Remark 2.6. Thus the natural homomorphism

$$\operatorname{Pic}\left(\bigcap_{i\in I} X_i\right) \to \operatorname{Pic}\left(\sum_{j\notin I}\left(X_j|_{(\bigcap_{i\in I} X_i)}\right)\right)$$

is injective by Theorem 1.8 (we note that $|\{1,\ldots,m\}\setminus I| = m-(k-1) \geq 2$). Thus the natural homomorphism

$$\operatorname{Pic}\left(\bigcap_{i\in I} X_i\right) \to \bigoplus_{j\notin I} \operatorname{Pic}\left(X_j \cap \bigcap_{i\in I} X_i\right)$$

is also injective by Proposition 2.7. We know that $\mathcal{L}|_{(X_j\cap\bigcap_{i\in I} X_i)}$ is trivial by the assumption. Therefore $\mathcal{L}|_{\bigcap_{i\in I} X_i}$ is also trivial. □

Theorem 3.11. (i) *Let $\mathcal{X}$ be an snc Fano variety, let $\mathcal{X} = \bigcup_{1\leq i\leq m} X_i$ be its decomposition into irreducible components and let $Z := X_1 \cap \cdots \cap X_m$. Then $\rho(\mathcal{X}) \leq \rho(Z) + m$. Moreover, if $\iota(\mathcal{X}) \geq 2$, then $\rho(\mathcal{X}) \leq \rho(Z)$ holds.*

(ii) *Let (X, D) be a log Fano manifold, let $D_1, \ldots, D_m$ be the irreducible components of D and let $Z := D_1 \cap \cdots \cap D_m$. Then $\rho(X) \leq \rho(Z) + m$. Moreover, if $\iota(X, D) \geq 2$, then $\rho(X) \leq \rho(Z)$ holds.*

Proof. (i) For any snc Fano variety $\mathcal{X}$, the rank of the Picard group $\operatorname{rank}(\operatorname{Pic}(\mathcal{X}))$ is equal to the Picard number $\rho(\mathcal{X})$. It is easily shown since $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$

(see [Fjn09, Corollary 2.26]). We can assume $m \geq 2$. By Lemma 3.10, the natural homomorphism

$$\mathrm{Pic}(\mathcal{X}) \to \bigoplus_{i=1}^{m} \mathrm{Pic}(X_1 \cap \cdots \check{X}_i \cdots \cap X_m)$$

is injective. We know that the pair $(X_1 \cap \cdots \check{X}_i \cdots \cap X_m, Z)$ is a log Fano manifold. Thus the rank of the kernel of the homomorphism

$$\mathrm{Pic}(X_1 \cap \cdots \check{X}_i \cdots \cap X_m) \to \mathrm{Pic}(Z)$$

is at most one by Theorem 1.8. Therefore the rank of the kernel of the homomorphism

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(Z)$$

is at most m. Moreover, if $\iota(\mathcal{X}) \geq 2$, then the assertion is trivial by Corollary 3.9 (2) and Remark 2.6.

(ii) If $m = 1$, then the assertion is trivial by Theorem 1.8. We can assume $m \geq 2$. We know that the natural homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(D)$$

is injective by Theorem 1.8. Hence the assertion follows from (i). □

The following is an easy consequence of Theorem 3.11.

Corollary 3.12. *For any $n \in \mathbb{Z}_{>0}$, there exists $p(n) \in \mathbb{Z}_{>0}$ that satisfies the following conditions.*

(i) *For any n-dimensional snc Fano variety $\mathcal{X}$, one has $\rho(\mathcal{X}) \leq p(n)$.*
(ii) *For any n-dimensional log Fano manifold (X, D), one has $\rho(X) \leq p(n)$.*

Proof. It is obvious by Theorem 3.11 and [KMM92, Theorem 0.2]. We note that the number of the irreducible components of $\mathcal{X}$ (resp. D) is at most $n+1$ (resp. n) by [Fjt12, Theorem 2.20 (2)]. □

Corollary 3.13. *For any four-dimensional log Fano manifold (X, D) with $D \neq 0$, the Picard number $\rho(X)$ of X satisfies $\rho(X) \leq 11$.*

Proof. This is a direct consequence of Theorem 3.11 and [MM81]. □

4. Application to the Mukai conjecture

In this section, we show that the Mukai conjecture (resp. the generalized Mukai conjecture) implies the log Mukai conjecture (resp. the log generalized Mukai conjecture). First, we see an important example of $(r\rho - \rho + 1)$-dimensional log Fano manifold of log Fano index r.

Example 4.1 (Type $(\rho, r; m_1, \dots, m_{\rho-1})$). Fix r, $\rho \geq 2$. Let $D \subset X$ be

$$\begin{aligned} X &:= \mathbb{P}_{(\mathbb{P}^{r-1})^{\rho-1}}(\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}^{\oplus r} \oplus \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(m_1, \dots, m_{\rho-1})) \\ D &:= \mathbb{P}_{(\mathbb{P}^{r-1})^{\rho-1}}(\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}^{\oplus r}) \end{aligned}$$

with $m_1 \dots, m_{\rho-1} \geq 0$, where the embedding $D \subset X$ is obtained by the canonical projection

$$\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}^{\oplus r} \oplus \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(m_1, \dots, m_{\rho-1}) \to \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}^{\oplus r}.$$

Then we have $\mathcal{O}_X(-K_X) \simeq p^*\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(r - m_1, \dots, r - m_{\rho-1}) \otimes \mathcal{O}_{\mathbb{P}}(r+1)$ and $\mathcal{O}_X(D) \simeq p^*\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(-m_1, \dots, -m_{\rho-1}) \otimes \mathcal{O}_{\mathbb{P}}(1)$, where $p\colon X \to (\mathbb{P}^{r-1})^{\rho-1}$ is the projection and

$$\mathcal{O}_{\mathbb{P}}(1) := \mathcal{O}_{\mathbb{P}_{(\mathbb{P}^{r-1})^{\rho-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1, \dots, m_{\rho-1}))}(1).$$

It is easy to show that the invertible sheaf $p^*\mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(1, \dots, 1) \otimes \mathcal{O}_{\mathbb{P}}(1)$ is ample. Hence (X, D) is an $(r\rho - \rho + 1)$-dimensional log Fano manifold with $r(X, D) = \iota(X, D) = r$.

We show now that the pair (X, D) in Example 4.1 is the only example of $(r\rho - \rho + 1)$-dimensional log Fano manifold with $D \neq 0$, $\rho(X) \geq \rho$ and $r(X, D) = r$ if we assume the low-dimensional Mukai conjecture.

Lemma 4.2. *Let r, $\rho \geq 2$. Consider a $\mathbb{P}^r$-bundle $\pi\colon X \to (\mathbb{P}^{r-1})^{\rho-1}$ and a divisor $D \subset X$ such that $D = (\mathbb{P}^{r-1})^{\rho}$ and the restriction is the projection morphism $\pi|_D = p_{1,\dots,\rho-1}\colon D = (\mathbb{P}^{r-1})^{\rho} \to (\mathbb{P}^{r-1})^{\rho-1}$ and is a $\mathbb{P}^{r-1}$-subbundle of π. If (X, D) is a log Fano manifold with $\iota(X, D) \geq r$, then (X, D) is isomorphic to the pair in Example 4.1 (for some $m_1, \dots, m_{\rho-1} \in \mathbb{Z}_{\geq 0}$).*

Proof. We can write the normal sheaf as $\mathcal{N}_{D/X} = \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho}}(-m_1 \dots, -m_{\rho-1}, 1)$ such that $m_1, \dots, m_{\rho-1} \in \mathbb{Z}$.

Claim 4.3. *We have $m_1, \dots, m_{\rho-1} \geq 0$.*

Proof of Claim 4.3. It is enough to show $m_1 \geq 0$. Let $P = \mathbb{P}^{r-1}$ be a general fiber of the projection $p_{2,\dots,\rho-1}\colon (\mathbb{P}^{r-1})^{\rho-1} \to (\mathbb{P}^{r-1})^{\rho-2}$ and let $X_P := \pi^{-1}(P)$, $\pi_P := \pi|_{X_P}\colon X_P \to P$ and $D_P := X_P \cap D$. Then (X_P, D_P) is also a log Fano manifold with $\iota(X_P, D_P) \geq \iota(X, D) \geq r$, the morphism π_P is a $\mathbb{P}^r$-bundle, $D_P = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$, the restriction morphism $(\pi_P)|_{D_P}\colon \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the first projection and a $\mathbb{P}^{r-1}$-subbundle of π_P. We also note that $\mathcal{N}_{D_P/X_P} \simeq \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}}(-m_1, 1)$. Hence $X_P \simeq \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}_{\mathbb{P}^{r-1}}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(m))$ with $m \geq 0$ and $D_P \simeq \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}_{\mathbb{P}^{r-1}}^{\oplus r})$, where the embedding is obtained by the canonical projection

$$\mathcal{O}_{\mathbb{P}^{r-1}}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(m) \to \mathcal{O}_{\mathbb{P}^{r-1}}^{\oplus r}$$

under the isomorphism, by [Fjt12, Theorem 4.3]. Thus we can show that $\mathcal{N}_{D_P/X_P} \simeq \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}}(-m, 1)$. Therefore we have $m_1 = m \geq 0$. □

The exact sequence

$$0 \to \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}} \to \pi_*\mathcal{O}_X(D) \to (\pi|_D)_*\mathcal{N}_{D/X} \to 0$$

in [Fjt12, Lemma 2.25 (i)] splits since we know that

$$(\pi|_D)_*\mathcal{N}_{D/X} \simeq \mathcal{O}_{(\mathbb{P}^{r-1})^{\rho-1}}(-m_1, \dots, -m_{\rho-1})^{\oplus r}$$

by [Fjt12, Lemma 2.28 (1)] and by Claim 4.3. Therefore we have proved Lemma 4.2 by [Fjt12, Lemma 2.25 (ii)]. □

Theorem 4.4. *Fix n, $\rho \geq 2$. Conjectures $\mathrm{M}_\rho^{n'}$ for all $n' \leq n$ (resp. Conjectures $\mathrm{GM}_\rho^{n'}$ for all $n' \leq n$) imply Conjecture LM_ρ^{n+1} (resp. Conjecture LGM_ρ^{n+1}).*

Proof. We only prove that Conjectures $\mathrm{GM}_\rho^{n'}$ for all $n' \leq n$ imply Conjecture LGM_ρ^{n+1}; the proof of the other assertion is essentially same.

Let (X, D) be an $(n+1)$-dimensional log Fano manifold with $D \neq 0$ such that $\rho(X) \geq \rho$ and $\iota := \iota(X, D) \geq (n+\rho)/\rho$, where n, $\rho \geq 2$. Let $D = \sum_{i=1}^m D_i$ be the decomposition of D into irreducible components and let $Z := \bigcap_{i=1}^m D_i$. Then Z is an $(n+1-m)$-dimensional Fano manifold with $\iota(Z) \geq \iota$. We know by Theorem 3.11 (i) that $\rho(Z) \geq \rho(X) \geq \rho$ since $\iota \geq 2$ holds. We note that

$$\iota \geq \frac{(n-1+m)+\rho}{\rho} \geq \frac{(n-1+m)+\rho-(m-1)}{\rho}.$$

Hence we can apply Conjecture GM_ρ^{n-1+m} for Z; we have $\rho(X) = \rho$, $\iota = (n+\rho)/\rho$, $m = 1$ and $Z = D \simeq (\mathbb{P}^{\iota-1})^\rho$. We can assume $D = (\mathbb{P}^{\iota-1})^\rho$.

We run a $(-D)$-MMP which is also a (K_X+D)-MMP as in Notation 3.4. The restriction morphism $\pi_0|_D\colon D \to \pi(D)$ to its image is an algebraic space and is not a finite morphism by Propositions 3.6 (2) and (3). Thus $\dim \pi(D) < n$ since $D \simeq (\mathbb{P}^{\iota-1})^\rho$. Hence $k = 0$, that is, $\pi_0\colon X \to Y^0$ is of fiber type contraction morphism, by Proposition 3.6 (1). We can assume that $Y^0 = (\mathbb{P}^{\iota-1})^{\rho-1}$ and the restriction morphism $\pi_0|_D\colon D \to Y^0$ is equal to the projection morphism $p_{1,\dots,\rho-1}\colon (\mathbb{P}^{\iota-1})^\rho \to (\mathbb{P}^{\iota-1})^{\rho-1}$ since $\rho(Y^0) = \rho - 1$ and $\pi_0(D) = Y^0$ holds by Proposition 3.6 (1). Let $[C] \in R^0$ be a minimal rational curve in R^0 on X. Then

$$\begin{aligned} \iota - 1 &= \dim(\pi_0^{-1}(y) \cap D) \geq \dim \pi_0^{-1}(y) - 1 \\ &\geq (-K_X \cdot C) - 2 = (-(K_X + D) \cdot C) + (D \cdot C) - 2 \geq \iota - 1 \end{aligned}$$

for any closed point $y \in Y^0$ by Wiśniewski's inequality [Wiś91] (see also [Fjt12, Theorem 2.29]). Thus we have $(-K_X \cdot C) = \iota + 1$, $(D \cdot C) = 1$ and $\dim \pi_0^{-1}(y) = \iota$ for any closed point $y \in Y^0$. Therefore the morphism $\pi_0\colon X \to Y^0$ is a $\mathbb{P}^\iota$-bundle and the restriction morphism $\pi_0|_D\colon D \to Y^0$ is a $\mathbb{P}^{\iota-1}$-subbundle of π_0 by [Fjt87, Lemma 2.12]. Hence Conjecture LGM_ρ^{n+1} holds by Lemma 4.2. □

Corollary 4.5. *Conjecture* LGM^n_ρ *is true if* (1) $\rho \leq 3$, *or* (2) $n \leq 6$.

Proof. It is an immediate corollary of Theorem 4.4 and Remark 1.7 (v). □

Theorem 4.6. *Fix* n, m, $\rho \in \mathbb{Z}_{>0}$. *Assume that the Mukai conjecture 1.1 (resp. the generalized Mukai conjecture 1.2) is true for any* $(n-m)$*-dimensional Fano manifold. Then for any* n*-dimensional log Fano manifold* (X, D) *such that the number of irreducible component of* D *is* m, *we have the inequality*

$$\rho(X)(r(X,D)-1) \leq n-m \quad (\textit{resp. } \rho(X)(\iota(X,D)-1) \leq n-m).$$

Proof. This is a straightforward consequence of Remark 2.6 and Theorem 3.11. □

Remark 4.7. When $m = 2$ under the notation and assumptions of Theorem 4.6, the author has classified the case where $\rho(X)(\iota(X,D)-1) = n-m$, whose result however consists of too many cases to be included here.

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